CS234r: Markets for Networks and Crowds
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Problem Set #1
Due: **Sept. 22, 2017**

Answer 5 of the following 6 questions.

1. (20 points) This question tests your basic comprehension of the lectures.

   (a) (10 points) **Market equilibria**. For each of the following combinatorial markets give a market equilibrium or prove that none exists. (Note: $V(\emptyset) = 0$ in all valuation functions.)

   - Items: $\{a, b\}$, agents: $\{1, 2\}$.
     $V_1(\{a, b\}) = 10$, $V_1(\{a\}) = V_1(\{b\}) = 3$
     $V_2(\{a, b\}) = V_2(\{a\}) = 6$, $V_2(\{b\}) = 3$
   - Items: $\{a, b, c, d\}$, agents: $\{1, 2\}$.
     $V_1(S) = \sqrt{|S|}$ for all S
     $V_2(S) = 1 + \frac{1}{2}|S|$ for all $S \neq \emptyset$
   - Items: $\{a, b, c, d\}$, agents: $\{1, 2\}$.
     $V_1(S) = \frac{1}{2}$ for all $S \neq \emptyset$
     $V_2(S) = \max\{|S|, 3\}$ for all $S \neq \emptyset$

   (b) (10 points) **Two-sided matching**. Consider the following matching market: there are 4 men and 4 women. The preferences are as follows.

m_1	$w_1 \succ w_2 \succ w_3 \succ w_4$
m_2	$w_2 \succ w_1 \succ w_4 \succ w_3$
m_3	$w_3 \succ w_4 \succ w_1 \succ w_2$
m_4	$w_4 \succ w_3 \succ w_2 \succ w_1$
w_1	$m_4 \succ m_3 \succ m_2 \succ m_1$
w_2	$m_3 \succ m_4 \succ m_1 \succ m_2$
w_3	$m_2 \succ m_1 \succ m_4 \succ m_3$
w_4	$m_1 \succ m_2 \succ m_3 \succ m_4$

   - List all 10 stable matchings. Call this set of stable matchings M.
   - For each agent x, let $S(x) = \{y : \mu(x) = y, \mu \in M\}$ be the multiset of his/her partners in the stable matchings you found above (note S should have 10 elements). Draw a table with 8 rows and 10 columns. In the i'th row, $1 \leq i \leq 4$, list the elements of $S(m_i)$ in *increasing order* of $\succ_{m_i}$ (so man m_i's favorite partner from $S(m_i)$ is in the first column). In the i'th row, $5 \leq i \leq 8$, list the elements of $S(w_i)$ in *decreasing order* of $\succ_{w_i}$ (so woman w_i's favorite partner from $S(w_i)$ is in the last column). What do you notice about the columns of this table? Hint: look at your answer to the first part.

- Imagine you are designing a two-sided matching market and need to choose a stable matching deterministically. Pick three different matchings from M and compare their potential benefits and drawbacks. **Provide a written description** (1 paragraph).

2. (20 points) In this question we will explore incentives in markets. Recall the following market model discussed in class: there is a single type of good being traded, n buyers, and m sellers. Buyer i has value $v_i \geq 0$ for receiving a copy of the good, and seller j can supply one copy of the good at production cost $c_j \geq 0$. A market mechanism asks the buyers and sellers to report their values and costs, then decides which buyers receive goods at what prices, and which sellers produce goods and how much they are paid.

   (a) (2 points) Show that the set of equilibrium prices in this market forms an interval. We'll call this interval $[\underline{p}, \overline{p}]$.

   (b) (3 points) Consider the following mechanism: compute the maximum equilibrium price $\overline{p}$ for the reported values and costs, and implement the market equilibrium corresponding to price $\overline{p}$. (That is, buyers and sellers make the socially efficient trades, each buyer that gets an item pays $\overline{p}$, and each seller that produces an item is paid $\overline{p}$.) Give an example showing that this mechanism is not strategyproof for the buyers.

   (c) (5 points) Consider a variant of the mechanism above that still implements the socially efficient set of trades, but each buyer that gets an item pays $\underline{p}$ and each seller that produces an item is paid $\overline{p}$. Prove that this mechanism is strategyproof for both buyers and sellers. Give an example showing that the mechanism might run a deficit, meaning that the total payment to sellers is larger than the total payment from the buyers.

   (d) (10 points) Suppose M is a mechanism that is strategyproof for buyers and sellers, implements the socially efficient set of trades, and always gives each participant non-negative utility. Prove that M must sometimes run a deficit. Hint: consider a setting with just one seller and one buyer. What do the assumptions imply about the payments when trade occurs?

3. (20 points) Consider the following Top Trading Cycles (TTC) mechanism for two-sided matching markets with complete strict preferences. (See Algorithm 1 below.) Each man points to his favorite woman, each woman points to her favorite man. While there is a cycle in the resulting graph, pick an arbitrary one and match each man in that cycle to his favorite woman in that cycle. Remove the cycles and repeat.

Let nodes $V \leftarrow$\{agents\};
Let directed edges $E \leftarrow \{(u,v) \in VxV : \forall v' \in V, v \succ_u v'\}$, i.e., v is favorite match of u;
Let directed graph $G \leftarrow (V, E)$;
$\mu \leftarrow \emptyset$;
while *there is a directed cycle in* G **do**
  foreach *directed cycle* $c = (u_1, \ldots, u_{2k})$ *in* G **do**
    if u_1 *is a man* **then**
      $\mu \leftarrow \mu \cup \{(u_1, u_2), (u_3, u_4), \ldots, (u_{2k-1}, u_{2k})\}$;
    else
      $\mu \leftarrow \mu \cup \{(u_2, u_3), (u_4, u_5), \ldots, (u_{2k}, u_1)\}$;
    end
  end
  $V \leftarrow V \setminus \{u : (u,v) \in \mu \vee (v,u) \in \mu\}$, i.e., the set of remaining unmatched agents;
  $E \leftarrow \{(u,v) \in VxV : \forall v' \in V, v \succ_u v'\}$, i.e., v is favorite match of u among remainig agents;
  $G \leftarrow (V, E)$;
end
return μ

Algorithm 1: The top trading cycles (TTC) algorithm with input $\succ_a$ for each agent.

(a) (2 points) Give an example with 3 men and 3 women where the TTC mechanism returns an unstable matching. Use the same example to show that the TTC mechanism is not strategyproof for the women.

(b) (6 points) Prove that the matching μ output by the TTC mechanism is pareto efficient. Hint: use induction on the set of men matched in each iteration of the algorithm.

(c) (12 points) Prove that the TTC mechanism is strategyproof for men. Hint: show that men never lose options, i.e., the set of women connected to a man in a directed path only grows until he is matched.

4. (20 points) You are in charge of designing a computer-aided market for a subsistence farmers in an African country. Small farms are scattered over a wide region, and farmers grow a variety of crops. These farms can be difficult to get to, and harvest times can be unpredictable. These farmers wish to sell their crops when they are ready, but the produce markets are in the big city and the farmers may not have the means to transport their goods. There are traders with trucks who are willing to purchase loads of crops to re-sell in the city, if they can do so at a profit.

   Without some sort of intervention, the traders and farmers have trouble matching with each other to make trades. You would like to develop a phone-based app that helps this market. It will collect bids (reported values) from traders and asks (reported costs) from farmers, and match them up to trade with each other.

   In this problem we explore some of the design choices for building such an app. For each question, **provide a written description** (1-3 paragraphs each) explaining your

approach. It is more important to explain your reasoning than to try to find the "right" answer – in many cases, there is not a single best approach.

(a) (5 points) How will your app decide which traders to match with which farmers? What sort of information could you collect from users that would be helpful with making these choices?

(b) (5 points) When a match is made, you may want to suggest a price at which the trade will occur. How would you choose the trading price?

(c) (5 points) Our models in class assumed a static set of buyers and sellers, but in this application the bids and asks arrive over time. How frequently will your system attempt to propose new trades, and what are the benefits and drawbacks of your choice?

(d) (5 points) Your app is becoming popular. In addition to bids and asks, you start getting requests for "current trading price" quotes. These are from users who are interested in what the "right" price is for a certain crop that's trading in your system. How would you produce these quotes?

5. (20 points) Prepare a discussion of Lerner's *The economics of control*, chapter 2. Summarize the contribution and any models described in this chapter, discuss the key new ideas, and/or suggest some next steps that build upon the paper. Do you like the paper? What are the important features of the market(s) being studied? Are the examples and discussions insightful? What are the limitations of the paper?

6. (20 points) Prepare a discussion of Gale and Shapley's *College admissions and the stability of marriage*, chapter 2. Summarize the contribution and any models described in this chapter, discuss the key new ideas, and/or suggest some next steps that build upon the paper. Do you like the paper? What are the important features of the market(s) being studied? Are the examples and discussions insightful? What are the limitations of the paper?